

AN APPLICATION OF GLOBAL WEYL MODULES OF $\mathfrak{sl}_{n+1}[t]$ TO INVARIANT THEORY.

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ABSTRACT. We identify \mathfrak{sl}_{n+1} isotypical components of global Weyl modules with natural subspaces in a polynomial ring, and then apply the representation theory of current algebras to classical problems in invariant theory.

INTRODUCTION

Given a positive integer k , let A_k be the algebra of polynomials with complex coefficients in k variables t_1, \dots, t_k and let S_k be the symmetric group on k letters. The group S_k acts naturally on A_k by permuting the variables. It is a classical result that A_k when regarded as a module for (the algebra of invariants) $A_k^{S_k}$ is free of rank $k!$.

In this paper, we are interested in certain variations of this problem. We regard A_k as being graded by the non-negative integers in the standard way with each generator having grade one and note that $A_k^{S_k}$ is a graded subalgebra. Given a partition $\xi = \xi_1 \geq \xi_2 \geq \dots \geq \xi_{n+1} > 0$ of k we define polynomials $\mathbf{p}(\mathbf{r}) \in A_k$ indexed by elements in $\mathbf{Z}_+^{\xi_2} \times \dots \times \mathbf{Z}_+^{\xi_{n+1}}$, where \mathbf{Z}_+ is the set of non-negative integers. (These polynomials can be thought of as being associated to Young symmetrizers of tableaux of shape ξ where the filling in the first row is 1). For instance, when $\xi = k - \ell \geq \ell > 0$, the polynomials are indexed by elements of \mathbf{Z}_+^ℓ and are defined by,

$$\mathbf{p}(\mathbf{s}) = \sum_{\sigma \in S_\ell} \left(\prod_{i=1}^{\ell} (t_{2i-1}^{s_{\sigma(i)}} - t_{2i}^{s_{\sigma(i)}}) \right), \quad \mathbf{s} \in \mathbf{Z}_+^\ell. \quad (0.1)$$

Let $M_{k,\xi}$ be the $A_k^{S_k}$ -submodule of A_k generated by the elements $\mathbf{p}(\mathbf{r}) \in \mathbf{Z}_+^{\xi_2} \times \dots \times \mathbf{Z}_+^{\xi_{n+1}}$. We use the theory global Weyl modules of the current algebra of \mathfrak{sl}_{n+1} to prove that $M_{k,\xi}$ is free and has a graded basis where the number of elements of a given grade is the coefficient of the corresponding power of q in the Kostka polynomial $\kappa_{1^k, \xi^{tr}}$ and show that it can be identified with the multiplicity space of ξ in A_k .

The global Weyl modules we use are indexed by multiples of the first fundamental weight of \mathfrak{sl}_{n+1} . These modules are well-understood in the literature and many spanning sets and bases are known for them: a Poincaré–Birkhoff–Witt type spanning set, a Gelfand–Tsetlin basis ([7]) a global canonical bases ([13]). In fact our main result gives an explicit way to construct a $A_k^{S_k}$ -generating set or basis of $M_{k,\xi}$ from a spanning set or basis of the appropriate global Weyl

VC acknowledges support from the NSF, DMS-0901253.
SL is supported by grants RFBR-09-01-00239, RFBR-CNRS-09-02-93106, N.Sh.8462.2010.1 .

module. The special case described above corresponds to taking a Poincare–Birkhoff–Witt spanning set for the global Weyl modules.

For $\xi = \ell \geq \ell$, the result was first conjectured by Chari and Greenstein. It was motivated by their study [5], [6] of the homological properties of the category of finite-dimensional representations of the current algebra of \mathfrak{sl}_2 . In [1] the module $M_{k, k-\ell \geq \ell}$ is denoted as $\mathbf{M}_{k,\ell}$ and a more specific conjecture is given which identifies a natural basis for it and this conjecture remains open.

Our interest in this result stems from the fact that it plays a crucial role in [2] which establishes a BGG-type duality for locally finite-dimensional representations of the current algebra associated to \mathfrak{sl}_2 . In Section 1, we give an elementary formulation of the results of this paper without using the notion of global Weyl modules. The definitions and necessary results on global Weyl modules are recalled in Section 2. In Section 3 we prove our main result which makes the identification between the module $M_{k,\xi}$ and the appropriate space of invariants in the global Weyl modules.

Acknowledgements. We thank Shrawan Kumar for several discussions. This work was completed when both authors were visiting the Hausdorff Institute, Bonn, in connection with the trimester “On the interactions of Representation theory with Geometry and Combinatorics,” and the authors thank the organizers of the trimester for the invitation to participate in the program.

1. THE MODULES $M_{k,\xi}$

1.1. Throughout this note we let \mathbf{Z}_+ (resp. \mathbf{N}) denote the set of non-negative (resp. positive) integers and \mathbf{C} the set of complex numbers. For $k \in \mathbf{N}$, we let A_k be the polynomial ring in k -variables equipped with the standard grading. For $r \in \mathbf{Z}_+$ the r -th graded piece is denoted as $A_k[r]$ and is the space of homogenous polynomials of degree r . We shall use without further mention, the celebrated result of Quillen (see [14] for an exposition) that any projective module for a polynomial ring is free.

1.2. We begin with some standard results [9] on the representation theory of the symmetric group S_k on k letters. Let $\mathcal{P}[k]$ be the set of all partitions $\xi = \xi_1 \geq \cdots \geq \xi_{n+1} \geq 0$ of k . The irreducible representations of S_k are parametrized by elements of $\mathcal{P}[k]$. By abuse of notation, we shall denote by ξ both a partition of k and an irreducible representation of S_k in the corresponding isomorphism class and we set $d_\xi = \dim \xi$. Any representation V of S_k can be written as a sum of finite dimensional irreducible modules and we denote by V^ξ the isotypical component corresponding to $\xi \in \mathcal{P}[k]$, and we have $V = \bigoplus_{\xi \in \mathcal{P}[k]} V^\xi$ as S_k -modules. We shall also use then notation V^{S_r} to denote the isotypical component corresponding to the trivial representation. Let $\mathbf{C}[S_k]$ be the group algebra of S_k over \mathbf{C} and regard it as module for itself via right multiplication. Then

$$\mathbf{C}[S_k] = \bigoplus_{\xi \vdash k} \mathbf{C}[S_k]^\xi, \quad \dim \text{Hom}_{S_k}(\xi, \mathbf{C}[S_k]^\xi) = d_\xi.$$

The following is well-known but we isolate it as a Lemma since it is used frequently. Recall that if V and V' are two representations of S_k then one has the diagonal action of S_k on $V \otimes V'$.

Lemma. *We have, $\text{Hom}_{S_k}(\xi, \xi) \neq 0$ or equivalently that $(\xi \otimes \xi)^{S_k} \neq 0$. Suppose that V is any representation of S_k such that $V^\xi \neq 0$. Then any non-zero element $v \in (V^\xi \otimes \xi)^{S_k}$ can be written as a sum,*

$$v = \sum_{s=1}^{d_\xi} w_s \otimes v_s, \quad w_s \in V^\xi, \quad v_s \in \xi,$$

where the elements $\{v_s : 1 \leq s \leq d_\xi\}$ are a basis of ξ and the elements $\{w_s : 1 \leq s \leq d_\xi\}$ span an irreducible S_k -submodule of V^ξ (in particular the elements $\{w_s, 1 \leq s \leq d_\xi\}$ are linearly independent). \square

1.3. Consider the natural right action of S_k on the ring A_k given by permuting the variables. The action preserves the graded pieces of A_k and the subalgebra $A_k^{S_k}$ of invariants is a graded subalgebra of A_k . Regard A_k as a module over the ring $A_k^{S_k}$ via right multiplication and note that it commutes with the action of S_k . Hence for any partition ξ of k , the S_k -isotypical component A_k^ξ is also a module for $A_k^{S_k}$ and we have an isomorphism of $A_k^{S_k}$ and S_k -modules,

$$A_k \cong \bigoplus_{\xi \vdash k} A_k^\xi.$$

By a classical result of Chevalley one knows that $A_k^{S_k}$ is also a polynomial ring in k variables and if we denote by I_k the augmentation ideal in $A_k^{S_k}$, then one has an isomorphism of S_k -modules,

$$A_k/I_k A_k^{S_k} \cong \mathbf{C}[S_k].$$

It follows that A_k is a free module for $A_k^{S_k}$ of rank $k!$ and in fact that each isotypical component A_k^ξ is also a free $A_k^{S_k}$ -module of rank d_ξ^2 . The augmentation ideal I_k is graded and hence $\mathbf{C}[S_k]$ acquires a \mathbf{Z}_+ -grading. It is known that the graded multiplicity of the irreducible representation ξ is given by the Kostka polynomial $\kappa_{1^{tr}, \xi}$.

1.4. To state our first result fix a partition $\xi = \xi_1 \geq \dots \geq \xi_{n+1} > 0$ of k with $(n+1)$ parts and set $k_i = \xi_i - \xi_{i+1}$, $1 \leq i \leq n+1$ where we understand that $k_{n+1} = \xi_{n+1}$. We shall define a family of elements $\mathbf{p}(r)$ in A_k which are indexed by elements of the set $\mathbf{Z}_+^{\xi_2} \times \dots \times \mathbf{Z}_+^{\xi_{n+1}}$. It will be convenient to think of A_k as a tensor product of algebras

$$A_k = A_1^{\otimes k_1} \otimes A_2^{\otimes k_2} \otimes \dots \otimes A_{n+1}^{\otimes k_{n+1}},$$

and we shall also identify

$$\Phi_\xi : \mathbf{Z}_+^{\xi_2} \times \dots \times \mathbf{Z}_+^{\xi_{n+1}} \cong \mathbf{Z}_+^{k_2} \times (\mathbf{Z}_+^2)^{k_3} \times \dots \times (\mathbf{Z}_+^{n+1})^{k_{n+1}},$$

as follows. (We use the convention that \mathbf{Z}_+^0 is the emptyset.) An element on the left hand side can naturally be thought of as an upper triangular matrix of size $k \times \xi_2$ (the entries in the (r, s) -the place are defined to be zero if $1 \leq s \leq k_r$) and this maps to the transpose matrix and then we drop all the zero entries. For example, if $\xi = 3 \geq 3 \geq 1$, then the map is

$$((r_1, r_2, r_3), s_1) \longrightarrow (r_1, r_2, (r_3, s_1)),$$

while if $\xi = 2 \geq 2 \geq 2$, then the map is

$$((r_1, r_2), (s_1, s_2)) \rightarrow ((r_1, s_1), (r_2, s_2)).$$

For $2 \leq \ell \leq k$ and $1 \leq s \leq k_\ell$, let $\Phi_\xi^{\ell,s} : \mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}} \rightarrow (\mathbf{Z}_+^\ell)^{k_\ell} \rightarrow \mathbf{Z}_+^\ell$ be the projection onto the ℓ -th factor and then onto the s -th factor. Given $(r_1, \dots, r_\ell) \in \mathbf{Z}_+^\ell$ set

$$\mathbf{a}(r_1, \dots, r_\ell) = \sum_{\tau \in S_{\ell+1}} \text{sgn}(\tau) t_{\tau(1)}^{r_1} \cdots t_{\tau(\ell)}^{r_\ell} \in A_{\ell+1}.$$

Next, given $\mathbf{r} \in \mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}$, set

$$\mathbf{b}(\mathbf{r}) = 1^{\otimes k_1} \otimes \mathbf{a}(\Phi_\xi^{2,1}(\mathbf{r})) \otimes \cdots \otimes \mathbf{a}(\Phi_\xi^{2,k_2}(\mathbf{r})) \otimes \cdots \otimes \mathbf{a}(\Phi_\xi^{n+1,1}(\mathbf{r})) \otimes \cdots \otimes \mathbf{a}(\Phi_\xi^{n+1,k_{n+1}}(\mathbf{r})).$$

Finally, we set

$$\mathbf{p}(\mathbf{r}) = \sum_{\sigma \in \mathbf{S}_\xi} \mathbf{b}(\sigma(\mathbf{r})),$$

where \mathbf{S}_ξ is the group $S_{\xi_2} \times \cdots \times S_{\xi_{n+1}}$ acting naturally on $\mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}$.

1.5. We compute an example to illustrate the preceding definitions and to relate it to the modules in the introduction. Suppose that $n = 1$ and $\xi = k - m \geq m > 0$ so that $k_1 = k - 2m$, $k_2 = m$. The polynomials we are interested in are indexed by elements of $\mathbf{r} = (r_1, \dots, r_m) \in \mathbf{Z}_+^m$ and the polynomials live in $A_1^{\otimes k-m} \otimes A_2^{\otimes m}$. Since $\Phi_\xi^{2,\ell}(\mathbf{r}) = r_\ell$, we have

$$\mathbf{b}(\mathbf{r}) = 1^{k-m} \otimes \mathbf{a}(r_1) \otimes \cdots \otimes \mathbf{a}(r_m) \quad \mathbf{a}(r) = (t_1^r - t_2^r) \in A_2.$$

Hence we get

$$\mathbf{p}(\mathbf{r}) = 1^{\otimes(k-m)} \otimes \sum_{\sigma \in S_m} (t_1^{r_{\sigma(1)}} - t_2^{r_{\sigma(1)}}) \cdots (t_1^{r_{\sigma(m)}} - t_2^{r_{\sigma(m)}}).$$

1.6. We shall prove,

Theorem 1. Let $k \in \mathbf{N}$ and let $\xi = \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n+1} > 0$ be a partition of k into $(n+1)$ parts and let $k_i = \xi_i - \xi_{i+1}$. The $A_k^{S_k}$ -submodule of A_k (denoted $M_{k,\xi}$) generated by the elements $\mathbf{p}(\mathbf{r})$, $\mathbf{r} \in \mathbf{Z}_+^{\xi_2} \times \cdots \times \mathbf{Z}_+^{\xi_{n+1}}$, is graded free and the number of elements of grade s in the basis is given the coefficient of q^s in the Kostka polynomial $\kappa_{1^k, \xi^{tr}}$. Moreover, we have an isomorphism of graded S_k and $A_k^{S_k}$ -modules

$$A_k^\xi \cong M_{k,\xi} \otimes \xi,$$

where we regard $A_k^{S_k}$ on $M_{k,\xi}$ and S_k as acting on ξ . In particular, if I_k is the augmentation ideal in A_k , we have a \mathbf{Z}_+ -graded map

$$M_{k,\xi}/I_k M_{k,\xi} \cong \text{Hom}_{S_k}(\xi, \mathbf{C}[S_k]).$$

We deduce this theorem from the results of the next two sections .

2. REPRESENTATIONS OF $\mathfrak{sl}_{n+1}[t]$

In this section we first recall several standard results on the representation theory of \mathfrak{sl}_{n+1} which can be found in any standard book [9], [12] for instance. We then define a family of representations (the global Weyl modules) of the infinite-dimensional Lie algebra $\mathfrak{sl}_{n+1} \otimes \mathbf{C}[t]$ and recall some of their properties which were first established in [8] and [10].

2.1. Let \mathfrak{sl}_{n+1} be the Lie algebra of $(n+1) \times (n+1)$ -matrices matrices of trace zero and \mathfrak{h} (resp. \mathfrak{n}^+ , \mathfrak{n}^-) be the subalgebra of diagonal (resp. strictly upper, lower triangular) matrices. Let $e_{i,j}$ be the $(n+1) \times (n+1)$ -matrix with one in the (i,j) -th position and zero elsewhere. The elements

$$x_i^+ = e_{i,i+1}, \quad h_i = e_{i,i} - e_{i+1,i+1}, \quad x_i^- = e_{i+1,i}, \quad 1 \leq i \leq n,$$

generate \mathfrak{sl}_{n+1} as a Lie algebra. Let $\Omega : \mathfrak{sl}_{n+1} \rightarrow \mathfrak{sl}_{n+1}$ be the anti-automorphism satisfying

$$\Omega(x_i^\pm) = x_i^\mp, \quad \Omega(h_i) = h_i. \quad (2.1)$$

2.2. Let \mathcal{P}^n be the set of all partitions with at most $(n+1)$ -parts and for $1 \leq i \leq n$, let $\omega_i \in \mathcal{P}^n$ be the partition 1^i . An element $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n+1} \geq 0$ of \mathcal{P}^n defines a finite-dimensional irreducible representation $V(\lambda)$ of \mathfrak{sl}_{n+1} and two partitions λ, λ' define isomorphic representations iff $\lambda_i - \lambda_{i+1} = \lambda'_i - \lambda'_{i+1}$ for all $1 \leq i \leq n$. The trivial representation of \mathfrak{sl}_{n+1} will be denoted as $V(\omega_0)$ and corresponds to taking the empty partition.

The module $V(\lambda)$ is generated by an element v_λ with defining relations:

$$\mathfrak{n}^+ v_\lambda = 0, \quad h_i v_\lambda = (\lambda_i - \lambda_{i+1}) v_\lambda, \quad (x_i^-)^{\lambda_i - \lambda_{i+1} + 1} v_\lambda = 0, \quad 1 \leq i \leq n.$$

Any irreducible finite-dimensional representation of \mathfrak{sl}_{n+1} is isomorphic to $V(\lambda)$ for some $\lambda \in \mathcal{P}^n$.

2.3. We say that a representation V of \mathfrak{sl}_{n+1} is locally finite-dimensional if it is a direct sum of finite-dimensional representations. By Weyl's theorem, this is equivalent to requiring that V is isomorphic to a direct sum of $V(\lambda)$, $\lambda \in \mathcal{P}^n$. The Lie algebra \mathfrak{h} acts diagonalizably on a locally finite-dimensional module V with integer eigenvalues, and we have

$$V = \bigoplus_{\mathbf{r} \in \mathbf{Z}^n} V_{\mathbf{r}}, \quad V_{\mathbf{r}} = \{v \in V : h_i v = r_i v\}, \quad \mathbf{r} \in \mathbf{Z}^n.$$

In the case when ξ is a partition, we also use the notation,

$$V_\xi = \{v \in V : h_i v = (\xi_i - \xi_{i+1}) v, \quad 1 \leq i \leq n\}, \quad V_\xi^{\mathfrak{n}^+} = \{v \in V_\xi : \mathfrak{n}^+ v = 0\}.$$

Note that $V(\lambda)^{\mathfrak{n}^+} = \mathbf{C} v_\lambda$ for all $\lambda \in \mathcal{P}^n$ and that in general, a locally finite-dimensional representation is generated as a \mathfrak{sl}_{n+1} -module by the spaces $V_\xi^{\mathfrak{n}^+}$, $\xi \in \mathcal{P}^n$. The following is elementary.

Lemma. *Let V be a locally finite-dimensional \mathfrak{sl}_{n+1} -module and let $\xi \in \mathcal{P}^n$. Then*

$$V_\xi = V_\xi^{\mathfrak{n}^+} \oplus (\mathfrak{n}^- V \cap V_\xi).$$

□

2.4. Recall that if U is any vector space there is a natural action of S_k on the k -fold tensor product $U^{\otimes k}$ which just permutes the factors. We shall need the following well-known result, the first two parts are elementary and the third part is the famous Schur–Weyl duality between representations of \mathfrak{sl}_{n+1} and those of S_k

Proposition. *Let $k, n \in \mathbf{N}$.*

(i) The module $V(\omega_1)$ is isomorphic to the natural representation \mathbf{C}^{n+1} of \mathfrak{sl}_{n+1} with standard basis e_1, \dots, e_{n+1} . Moreover the assignment $(e_i, e_j) = \delta_{i,j}$ defines a symmetric bilinear form $(\cdot, \cdot) : V(\omega_1) \otimes V(\omega_1) \rightarrow \mathbf{C}$ satisfying,

$$(xv, v') = (v, \Omega(x)v'), \quad v, v' \in V(\omega_1), \quad x \in \mathfrak{sl}_{n+1}.$$

(ii) For $1 \leq i \leq n+1$, we have

$$(V(\omega_1)^{\otimes i})_{\omega_i}^{\mathfrak{n}^+} = \mathbf{C}(e_1 \wedge e_2 \wedge \cdots \wedge e_i),$$

and an isomorphism of \mathfrak{sl}_{n+1} -modules $\bigwedge^i V(\omega_1) \cong V(\omega_i)$.

(iii) Let $k \in \mathbf{N}$. The natural right action of S_k on $V(\omega_1)^{\otimes k}$ commutes with the left action of \mathfrak{sl}_{n+1} and as a $(\mathfrak{sl}_{n+1}, S_k)$ -bimodule, we have,

$$V(\omega_1)^{\otimes k} \cong \bigoplus_{\xi \in \mathcal{P}^n \cap \mathcal{P}[k]} V(\xi) \otimes \xi,$$

where the sum is over all partitions of k with at most $(n+1)$ parts. Equivalently, the space $(V(\omega_1)^{\otimes k})_{\xi}^{\mathfrak{n}^+}$ is an irreducible S_k -submodule of $V(\omega_1)^{\otimes k}$ and we have an isomorphism of S_k -modules

$$(V(\omega_1)^{\otimes k})_{\xi}^{\mathfrak{n}^+} \cong \xi.$$

□

2.5. Denote by $(\cdot, \cdot)_k : V(\omega_1)^{\otimes k} \times V(\omega_1)^{\otimes k} \rightarrow \mathbf{C}$ the symmetric non-degenerate bilinear form given by extending linearly the assignment,

$$(e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k})_k = (e_{i_1}, e_{j_1}) \cdots (e_{i_k}, e_{j_k}),$$

where $i_s, j_s \in \{1, \dots, n+1\}$ for $1 \leq s \leq k$. The form is clearly S_k -invariant and moreover,

$$(x\mathbf{v}, \mathbf{v}')_k = (\mathbf{v}, \Omega(x)\mathbf{v}')_k, \quad \mathbf{v}, \mathbf{v}' \in V(\omega_1)^{\otimes k}, \quad x \in \mathfrak{g}.$$

In particular since $\Omega(h_i) = h_i$ for all $1 \leq i \leq n$, we have

$$(V(\omega_1)^{\otimes k})_{\mathbf{r}} \cdot (V(\omega_1)^{\otimes k})_{\mathbf{r}'} = 0, \quad \mathbf{r}, \mathbf{r}' \in \mathbf{Z}^n, \quad \mathbf{r} \neq \mathbf{r}',$$

and so the restriction of the form to $V(\omega_1)^{\otimes k}$ is non-degenerate.

Lemma. For all $\xi \in \mathcal{P}^n \cap \mathcal{P}[k]$ the restriction of the form $(\cdot, \cdot)_k$ to $(V(\omega_1)^{\otimes k})_{\xi}^{\mathfrak{n}^+} \times (V(\omega_1)^{\otimes k})_{\xi}^{\mathfrak{n}^+}$ is non-degenerate.

Proof. Let $\mathbf{v}, \mathbf{v}' \in V(\omega_1)^{\otimes k}_{\xi}^{\mathfrak{n}^+}$ be such that $(\mathbf{v}, \mathbf{v}')_k \neq 0$ and assume that $\mathfrak{n}^+ \mathbf{v} = 0$. By Lemma 2.3, we may write $\mathbf{v}' = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathfrak{n}^+ \mathbf{v}_1 = 0$ and $\mathbf{v}_2 = \sum_{i=1}^n e_{i+1,i} \mathbf{u}_i$ for some $\mathbf{u}_i \in V(\omega_1)^{\otimes k}$. Hence we get

$$(\mathbf{v}, \mathbf{v}_2)_k = (\mathbf{v}, \sum_{i=1}^n e_{i+1,i} \mathbf{u}_i)_k = (\sum_{i=1}^n (e_{i,i+1} \mathbf{v}, \mathbf{u}_i)_k = 0,$$

i.e., $(\mathbf{v}, \mathbf{v}')_k = (\mathbf{v}, \mathbf{v}_1)_k \neq 0$ and the Lemma is proved. □

2.6. From now on, we fix $k \in \mathbf{N}$ and a partition $\xi = \xi_1 \geq \cdots \geq \xi_{n+1} > 0$ of k . Let $k_i = \xi_i - \xi_{i+1}$ and set

$$\mathbf{e}(\xi) = e_1^{\otimes k_1} \otimes (e_1 \wedge e_2)^{\otimes k_2} \otimes \cdots \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{n+1})^{\otimes k_{n+1}} \in (V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}.$$

Let $S_k(\xi)$ be the isotropy subgroup of S_k which fixing $\mathbf{e}(\xi)$ and fix a set of distinct coset representatives σ_ℓ , $1 \leq \ell \leq d_\xi - 1$ with $\sigma_1 = \text{id}$. We note the following consequence of Proposition 2.4 and the discussion in Section 2.5.

Lemma. *The elements $\{\mathbf{e}(\xi)\sigma_\ell : 1 \leq \ell \leq d_\xi\}$ are a basis of $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$. Moreover there exists a dual basis $\{\mathbf{e}_\ell : 1 \leq \ell \leq d_\xi\}$ of $(V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+}$ satisfying*

$$(\mathbf{e}_j, \mathbf{e}(\xi)\sigma_m)_k = 0, \quad j \neq m, \quad (\mathbf{e}_m, \mathbf{e}(\xi)\sigma_m)_k = 1. \quad (2.2)$$

□

2.7. We now turn our attention to representations of $\mathfrak{sl}_{n+1}[t]$ and we begin with some general definitions. Given a complex Lie algebra \mathfrak{a} and an indeterminate t , let $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbf{C}[t]$ be the Lie algebra with commutator given by,

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg, \quad a, b \in \mathfrak{a}, \quad f, g \in \mathbf{C}[t].$$

We identify without further comment the Lie algebra \mathfrak{a} with the subalgebra $\mathfrak{a} \otimes 1$ of $\mathfrak{a}[t]$. Clearly, $\mathfrak{a}[t]$ has a natural \mathbf{Z}_+ -grading given by the powers of t and this also induces a \mathbf{Z}_+ -grading on $\mathbf{U}(\mathfrak{a}[t])$. A \mathbf{Z}_+ -graded bimodule for the pair $(\mathfrak{a}[t], A_k)$ is a complex vector space $V = \bigoplus_{s \in \mathbf{Z}_+} V_s$ which admits a left action of $\mathfrak{a}[t]$, a right action of A_k and both actions are compatible with the grading,

$$(\mathfrak{a} \otimes t^r)V[s] \subset V[s+r], \quad V[s]A_k[r] \subset V[s+r], \quad s, r \in \mathbf{Z}_+.$$

An elementary way to construct such modules is as follows: let U_s , $1 \leq s \leq k$ be \mathfrak{a} -modules and define on $(U_1 \otimes \cdots \otimes U_k \otimes A_k)$ the structure of a bimodule for the pair $(\mathfrak{a}[t], A_k)$ by:

$$(u_1 \otimes \cdots \otimes u_k \otimes a)b = u_1 \otimes \cdots \otimes u_k \otimes ab, \quad (2.3)$$

$$(x \otimes t^r)(u_1 \otimes \cdots \otimes u_k \otimes a) = \sum_{s=1}^k \left(\otimes_{j=1}^{s-1} u_j \right) \otimes xu_s \left(\otimes_{j=s+1}^k u_j \right) \otimes t_s^r a. \quad (2.4)$$

The grading on $(U_1 \otimes \cdots \otimes U_k \otimes A_k)$ is induced by the grading on A_k , i.e., for an integer r , the r^{th} -graded piece is $(U_1 \otimes \cdots \otimes U_k) \otimes A_k[r]$. These modules are clearly free as right A_k (and so also as right $A_k^{S_k}$) modules.

2.8. Consider the special case when all the U_i 's are the same, say $U_i = U$ for $1 \leq i \leq k$. The induced diagonal action of S_k on $U^{\otimes k} \otimes A_k$ commutes with the right action of A_k and hence for each $\xi \in \mathcal{P}^k$, the isotypical component $(U^{\otimes k} \otimes A_k)^\xi$ is a $A_k^{S_k}$ -bimodule and we have an isomorphism of $A_k^{S_k}$ -modules

$$U^{\otimes k} \otimes A_k \cong \bigoplus_{\xi \in \mathcal{P}^k} (U^{\otimes k} \otimes A_k)^\xi,$$

If $\dim U < \infty$ then $(U^{\otimes k} \otimes A_k)^\xi$ is a free $A_k^{S_k}$ -module of finite rank. The following is easily checked by using the explicit definition of the action of $\mathfrak{a}[t]$ given in (2.4).

Lemma. *The space $(U^{\otimes k} \otimes A_k)^{S_k}$ is a \mathbf{Z}_+ -graded $(\mathfrak{a}[t], A_k^{S_k})$ -submodule of $U^{\otimes k} \otimes A_k$. \square*

2.9. Apply the preceding construction to the case when $\mathfrak{a} = \mathfrak{sl}_{n+1}$ and $U = V(\omega_1)$. Define an A_k -valued bilinear form

$$\langle \cdot, \cdot \rangle : (V(\omega_1)^{\otimes k} \otimes A_k) \times (V(\omega_1)^{\otimes k} \otimes A_k) \rightarrow A_k,$$

by extending the assignment,

$$\langle \mathbf{v} \otimes f, \mathbf{v}' \otimes g \rangle = (\mathbf{v}, \mathbf{v}')_k f g,$$

where $\mathbf{v}, \mathbf{v}' \in V(\omega_1)^{\otimes k}$, and $f, g \in A_k$. Notice that the form is not S_k -invariant, however, we do have the following which is proved by a direct calculation.

Lemma. *For all $x \in \mathfrak{sl}_{n+1}$, $f \in \mathbf{C}[t]$, we have*

$$\langle (x \otimes f)(\mathbf{v} \otimes g), \mathbf{v}' \otimes g' \rangle = \langle (\mathbf{v} \otimes g, (\Omega(x) \otimes f)(\mathbf{v}' \otimes g')) \rangle,$$

where $\mathbf{v}, \mathbf{v}' \in (V(\omega_1)^{\otimes k})$ and $g, g' \in A_k$. Further,

$$\langle \cdot, \cdot \rangle : V(\omega_1)^{\otimes k} \otimes A_k \rightarrow A_k,$$

is a map of A_k -modules for all $\mathbf{v} \in V(\omega_1)^{\otimes k}$. \square

2.10. For $k \in \mathbf{Z}_+$, set

$$W(k) = \left(V(\omega_1)^{\otimes k} \otimes A_k \right)^{S_k}.$$

Since $W(k)$ is a locally finite-dimensional \mathfrak{sl}_{n+1} -module we can write

$$W(k) = \bigoplus_{\mathbf{r} \in \mathbf{Z}^n} W(k)_{\mathbf{r}}, \quad W(k)_{\xi} = W(k)_{\xi}^{\mathfrak{n}^+} \oplus (\mathfrak{n}^- W(k) \cap W(k)_{\xi}).$$

Since the action of \mathfrak{sl}_{n+1} on $W(k)$ commutes with the action of $A_k^{S_k}$, we see that $W(k)_{\mathbf{r}}$ and $W(k)_{\xi}^{\mathfrak{n}^+}$ are graded free right $A_k^{S_k}$ -module and in fact, we have

$$W(k)_{\mathbf{r}} = \left(V(\omega_1)_{\mathbf{r}}^{\otimes k} \otimes A_k \right)^{S_k}, \quad W(k)_{\xi}^{\mathfrak{n}^+} = \left((V(\omega_1)^{\otimes k})_{\xi}^{\mathfrak{n}^+} \otimes A_k \right)^{S_k}. \quad (2.5)$$

2.11. We shall need the following result. As usual we denote by $\mathbf{U}(\mathfrak{a})$ the universal enveloping algebra of the Lie algebra \mathfrak{a} .

Theorem 2. Let $k \in \mathbf{N}$.

- (i) The element $v_{\omega_1}^{\otimes k}$ generates $W(k)$ as a module for $\mathfrak{sl}_{n+1}[t]$ and moreover,

$$W(k) = \mathbf{U}(\mathfrak{n}^-[t])(v_{\omega_1}^{\otimes k} \otimes A_k^{S_k}).$$

- (ii) For $\xi \in \mathcal{P}^n$, the $A_k^{S_k}$ -module $W(k)_{\xi}^{\mathfrak{n}^+}$ is graded free and the number of elements in a basis with grade s is the coefficient of q^s in the Kostka polynomial $\kappa_{1^k, \xi^{tr}}$. \square

Part (i) of the theorem was proved in [8] for \mathfrak{sl}_2 and in [10] for \mathfrak{sl}_{n+1} . Recall that I_k is the augmentation ideal in $A_k^{S_k}$ and consider the graded $\mathfrak{sl}_{n+1}[t]$ -module, $(W(k)/I_k W(k))$. It was shown in [7] that the subspace $(W(k)/I_k W(k))\mathfrak{n}^+_\xi$ has a graded basis and the number of elements in the basis of grade s is the coefficient of q^s in the Kostka polynomial $\kappa_{1^k, \xi^{tr}}$. Part (ii) follows since $W(k)_\xi^{\mathfrak{n}^+}$ is graded free as an $A_k^{S_k}$ -module and we have an isomorphism of graded spaces,

$$W(k)_\xi^{\mathfrak{n}^+}/I_k W(k)_\xi^{\mathfrak{n}^+} \cong (W(k)/I_k W(k))_\xi^{\mathfrak{n}^+}.$$

2.12. The modules $W(k)$ are special examples of a family of infinite-dimensional modules called global Weyl modules which are defined for arbitrary simple Lie algebras. For \mathfrak{sl}_{n+1} it is proved in [3] that the global Weyl modules can be realized as sitting inside a suitable space of invariants. However, except in the special case we consider in this paper, the global Weyl modules are strictly smaller than the space of invariants.

3. THE MAIN RESULT AND PROOF OF THEOREM 1

3.1. As before we fix $k \in \mathbf{N}$, $\xi = \xi_1 \geq \xi_2 \geq \dots \geq \xi_{n+1} > 0$ and set $k_i = \xi_i - \xi_{i+1}$, $1 \leq i \leq n+1$ and we freely use the notation of the earlier sections. Theorem 1 is clearly a consequence Theorem 2 and the following result.

Theorem 3. The restriction of the map $\langle \cdot, \mathbf{e}(\xi) \rangle : V(\omega_1)^{\otimes k} \otimes k \rightarrow A_k$ gives an isomorphism of $A_k^{S_k}$ -modules $W(k)_\xi^{\mathfrak{n}^+} \rightarrow M_{k,\xi}$. Moreover, for all $\sigma \in S_k$ we have

$$M_{k,\xi}\sigma = \langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi)\sigma \rangle, \quad (3.1)$$

$$\langle W(k)_\xi^{\mathfrak{n}^+}, (V(\omega_1)^{\otimes k})_\xi^{\mathfrak{n}^+} \rangle \cong \bigoplus_{s=1}^{d_\xi} M_{k,\xi}\sigma_s \cong A_k^\xi, \quad (3.2)$$

as S_k and $A_k^{S_k}$ -modules.

Proof. Since $\langle \cdot, \mathbf{e}(\xi) \rangle : V(\omega_1)^k \otimes A_k \rightarrow A_k$ is a map of A_k -modules, it follows that the restriction to $W(k)_\xi^{\mathfrak{n}^+}$ is a map of $A_k^{S_k}$ -modules. Using (2.5) we have

$$W(k)_\xi^{\mathfrak{n}^+} = ((V(\omega_1)^k)_\xi^{\mathfrak{n}^+} \otimes A_k)^{S_k},$$

and hence we may write any non-zero element $\mathbf{v} \in W(k)_\xi^{\mathfrak{n}^+}$ as $\mathbf{v} = \sum_{s=1}^{d_\xi} \mathbf{e}_s \otimes g_s$, where $g_s \in A_k$ and $\{\mathbf{e}_s : 1 \leq s \leq d_\xi\}$ is the basis defined in Section 2.6. By Lemma 1.2 we see that $g_1 \neq 0$ and using (2.2) we now see that

$$\langle \mathbf{v}, \mathbf{e}(\xi) \rangle = g_1 \neq 0,$$

which proves that the map $W(k)_\xi^{\mathfrak{n}^+} \rightarrow A_k$ is injective.

To prove that

$$\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi) \rangle = M_{k,\xi},$$

we prove simultaneously that the elements $\mathbf{p}(\mathbf{r}) \in \langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi) \rangle$ and that they generate the $\langle W(k)_\xi^{\mathfrak{n}^+}, \mathbf{e}(\xi) \rangle$ as an $A_k^{S_k}$ -module. By Theorem 2 we may write an element $\mathbf{v} \in W(k)_\xi$ as

$$\mathbf{v} = \sum_{s=1}^m y_s (v_{\omega_1}^{\otimes k} \otimes a_s), \quad y_s \in \mathbf{U}(\mathfrak{n}^-[t]), \quad a_s \in A_k^{S_k}.$$

Since $x_{j,k} v_{\omega_1} = 0$, $k \neq 1$, it follows from the Poincare–Birkhoff–Witt theorem that we may assume that y_s is a product of elements from the set $\{x_{j,1} \otimes t^r : 2 \leq j \leq n+1, r \in \mathbf{Z}_+\}$. Moreover since $[x_{j,1}, x_{\ell,1}] = 0$ for $2 \leq j, \ell \leq n+1$ we can write

$$y_s = \prod_{j=2}^{n+1} (e_{j,1} \otimes t^{r_{j,1}}) \cdots (e_{j,1} \otimes t^{r_{j,p_j}}),$$

for some $r_{j,p} \in \mathbf{Z}_+$, $2 \leq j \leq n+1$ and $1 \leq p_j \leq d_{\xi_j}$. If $r_{j,p} = 0$ for some j , i.e., if $y_s \in \mathfrak{n}^- \mathbf{U}(\mathfrak{n}^-)[t]$, then since

$$\langle \mathfrak{n}^- bu(\mathfrak{n}^-)[t] v_{\omega_1}^{\otimes k}, boe(\xi) \rangle = \langle \mathbf{U}(\mathfrak{n}^-)[t] v_{\omega_1}^{\otimes k}, \mathfrak{n}^+ boe(\xi) \rangle = 0,$$

we see that

$$(y_s v_{\omega_1}^{\otimes k}, \mathbf{e}(\xi)) = 0. \quad (3.3)$$

If $r_{j,p} > 0$ for all j then,

$$\langle y_s v_{\omega_1}^{\otimes k}, \mathbf{e}(\xi) \rangle = \langle v_{\omega_1}^{\otimes k}, \prod_{j=2}^{n+1} (e_{1,j} \otimes t^{r_{j,1}}) \cdots (e_{1,j} \otimes t^{r_{j,p_j}}) \mathbf{e}(\xi) \rangle = \mathbf{p}(\mathbf{r}),$$

where a tedious but straightforward calculation shows that

$$\prod_{j=2}^{n+1} (e_{1,j} \otimes t^{r_{j,1}}) \cdots (e_{1,j} \otimes t^{r_{j,p_j}}) \mathbf{e}(\xi) = v_{\omega_1}^{\otimes k} \otimes \mathbf{p}(\mathbf{r}). \quad (3.4)$$

For instance in the case of \mathfrak{sl}_2 , one sees that

$$(e_{1,2} \otimes t^{r_1}) \otimes \cdots \otimes (e_{1,2} \otimes t^{r_\ell}) \left((e_1^{k-2\ell} \otimes (e_1 \wedge e_2)^{\otimes \ell}) \right) = e_1^{k-2\ell} \otimes (e_{1,2} \otimes t^{r_1}) \otimes \cdots \otimes (e_{1,2} \otimes t^{r_\ell}) (e_1 \wedge e_2)^{\otimes \ell}.$$

Now use the fact that

$$e_{1,2} \otimes t^r (e_1 \otimes e_2 - e_2 \otimes e_1) = (e_1 \otimes e_1 \otimes t_2^r - e_1 \otimes e_1 \otimes t_1^r) = (e_1 \otimes e_1) \otimes (t_2^r - t_1^r),$$

to complete the calculation.

A consequence of equations (3.3) and (3.4) is the following: if $y_s v_{\omega_1}^{\otimes k} \neq 0$, then $y_s v_{\omega_1}^{\otimes k}$ has non-zero projection onto $W(k)_\xi^{\mathfrak{n}^+}$. We have already noted that $\langle \mathfrak{n}^- W(k), \mathbf{e}(\xi) \rangle = 0$ and hence we have now proved both that $\mathbf{p}(\mathbf{r})$ is in the image of the map $\langle \ , \mathbf{e}(\xi) \rangle$ and that the image is generated as an $A_k^{S_k}$ -module by these elements.

To prove (3.1), let $\mathbf{a} \in M_{k,\xi}$ and choose $\mathbf{v} = \sum_{s=1}^\ell v_s \otimes f_s \in W(k)_\xi^{\mathfrak{n}^+}$ such that

$$\mathbf{a} = \langle \mathbf{v} \ , \mathbf{e}(\xi) \rangle = \sum_{s=1}^\ell (v_s, \mathbf{e}(\xi))_k f_s.$$

For $\sigma \in S_k$, we get by using the S_k -invariance of $(\ , \)_k$ that,

$$\begin{aligned} \mathbf{a}\sigma &= \sum_{s=1}^{\ell} (v_s\sigma, \mathbf{e}(\xi)\sigma)_k f_s\sigma = \left\langle \sum_{s=1}^{\ell} v_s\sigma \otimes f_s\sigma, \mathbf{e}(\xi)\sigma \right\rangle \\ &= \left\langle \left(\sum_{s=1}^{\ell} v_s \otimes f_s \right) \sigma, \mathbf{e}(\xi)\sigma \right\rangle = \langle \mathbf{v}\sigma, \mathbf{e}(\xi)\sigma \rangle = \langle \mathbf{v} \ , \ \mathbf{e}(\xi)\sigma \rangle, \end{aligned}$$

where the last equality is consequence of the fact that elements of $W(k)$ are S_k -invariant. This proves that

$$M_{k,\xi}\sigma = \langle W(k)_{\xi}^{\mathbf{n}^+}, \mathbf{e}(\xi)\sigma \rangle, \quad \sigma \in S_k,$$

and hence also that we have an isomorphism of S_k and $A_k^{S_k}$ modules,

$$\langle W(k)_{\xi}^{\mathbf{n}^+} \ , \ (V(\omega_1)^{\otimes k})_{\xi}^{\mathbf{n}^+} \rangle \cong \sum_{\sigma \in S_k} M_{k,\xi}\sigma.$$

We now prove that

$$\sum_{\sigma \in S_k} M_{k,\xi}\sigma = A^{\xi}.$$

To prove that the left hand side is contained in the right hand side, it suffices to prove that $M_{k,\xi} \subset A^{\xi}$. But this is clear since as we have observed before any $\mathbf{a} \in M_{k,\xi}$ may be written as,

$$\mathbf{a} = \langle \mathbf{v} \ , \ \mathbf{e}(\xi) \rangle = \sum_{s=1}^{\ell} (v_s, \mathbf{e}(\xi))_k f_s,$$

where

$$\mathbf{v} \in W(k)_{\xi}^{\mathbf{n}^+} = \left(V(\omega_1)^{\otimes A_k} \right)_{\xi}^{\mathbf{n}^+} \otimes A_k = \left(V(\omega_1)^{\otimes k} \right)_{\xi}^{\mathbf{n}^+} \otimes A_k^{S_k}.$$

This means we may assume that $f_s \in A^{\xi}$ for all $1 \leq s \leq \ell$ which proves that $\mathbf{a} \in A^{\xi}$.

For the reverse inclusion, suppose that $a \in A_k^{\xi}$ and assume that it generates an irreducible S_k -submodule N and let $a_1, \dots, a_{d_{\xi}}$ be a basis of N where $a_1 = a$. Then there exists a non-zero element

$$\mathbf{v} \in (V(\omega_1)^{\otimes k})_{\xi}^{\mathbf{n}^+} \otimes N \subset W(k)_{\xi}^{\mathbf{n}^+},$$

and by Lemma 1.2 we can write $\mathbf{v} = \sum_{s=1}^{d_{\xi}} \mathbf{v}_s \otimes a_s$ with $\mathbf{v}_s \neq 0$. In fact, the elements $\{\mathbf{v}_s : 1 \leq s \leq d_{\xi}\}$ are a basis for $(V(\omega_1)^{\otimes k})_{\xi}^{\mathbf{n}^+}$ and by Lemma 2.5 we may choose a dual basis $\{\mathbf{v}'_s : 1 \leq s \leq d_{\xi}\}$ and we find now that

$$\langle \mathbf{v}, \mathbf{v}'_1 \rangle = a_1 = a,$$

which proves that A^{ξ} is contained in M_{ξ} .

To complete the proof of the theorem, we must prove that $M_{k,\xi}\sigma$ is a summand of A^{ξ} as an $A_k^{S_k}$ -module. We have so far proved that $M_{k,\xi}\sigma$ is free of rank d_{ξ} and also that

$$A^{\xi} = \sum_{s=1}^{d_{\xi}} M_{k,\xi}\sigma_s.$$

Since A^ξ is a free module of rank d_ζ^2 , it follows that the sum is direct and the proof of the theorem is complete. \square

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